Quantum chaos in many-particle systems

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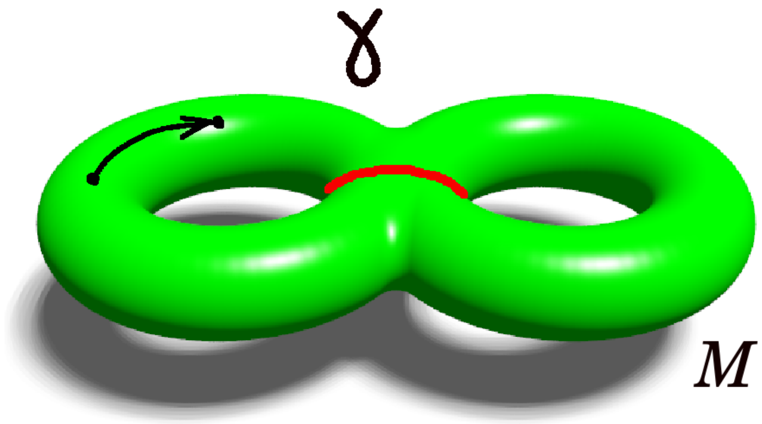
Outline of the talk

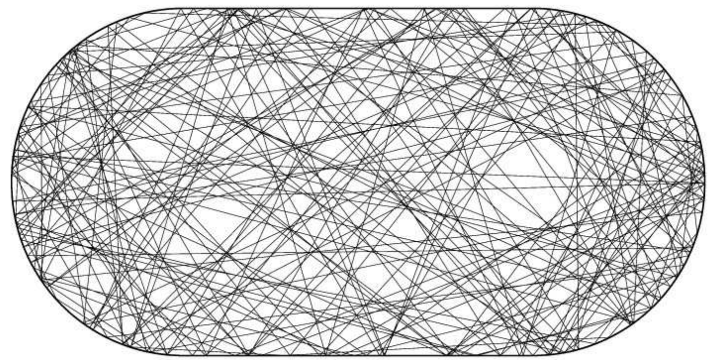
- “Single”-particle quantum chaos.
  Single (semiclassical) limit: $\hbar \to 0$

- Many-particle quantum chaos.
  Double limit: $N \to \infty$, $\hbar \to 0$

  B.G. $\&$ V. Al. Osipov, Nonlinearity 29 (2016)
  arXiv:1503.02676

Chaos & Spectral universality





Classical chaos: $\delta(t) \sim \delta(0)e^{\lambda t}$

Motivation

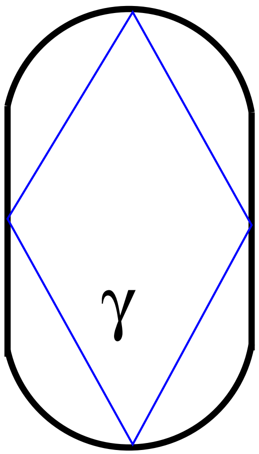
Quantum: $-\Delta\varphi_n = \lambda_n\varphi_n, \qquad \varphi_n \in L^2(M)$

BGS conjecture G.Casati, et al. 1980; O. Bohigas, et al. 1984: Correlations of $\{\lambda_n\}_{n=1}^{\infty}$ are universal, described by Random Matrix Ensembles from the same symmetry class

Semiclassical approach

Gutzwiller's trace formula:

$$\rho(E) = \sum_n \delta(E - E_n) \sim \underbrace{\bar{\rho}(E)}_{\color{blue}{Smooth}} + \underbrace{\Re \sum_{\gamma \in \mathrm{PO}} \mathcal{A}_\gamma \exp\left(\frac{i}{\hbar} S_\gamma(E)\right)}_{\color{red}{Oscillating}}$$



$\mathcal{A}_\gamma$ stability factor,
$S_\gamma(E)$ action of a **periodic orbit** γ

Number of periodic orbits grows exponentially with length

– No prediction on E_n from an individual γ
– **All** $\{\gamma\}$ **together** $\Longleftrightarrow$ **spectrum**

Two-point correlation function

$$R(\varepsilon) = \frac{1}{\bar{\rho}^2} \langle \rho(E + \varepsilon/\bar{\rho})\rho(E) \rangle_E - 1$$

$$K(\tau) = \int_{-\infty}^{+\infty} R(\varepsilon) e^{-2\pi i \tau \varepsilon} d\varepsilon \approx \text{ (Semiclassically)}$$

$$\approx \frac{1}{T_H^2} \left\langle \sum_{\gamma,\gamma'} \mathcal{A}_\gamma \mathcal{A}^*_{\gamma'} e^{\frac{i}{\hbar}(S_\gamma - S_{\gamma'})} \delta\left(\tau - \frac{(T_\gamma + T_{\gamma'})}{2T_H}\right) \right\rangle_E,$$

$T_\gamma, T_{\gamma'}$ are periods of γ, γ', $\quad T_H = 2\pi\hbar\bar{\rho}$ (Heisenberg time)

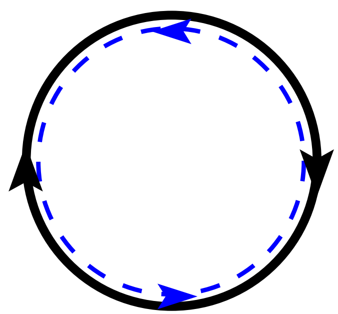
Spectral correlations $\Longleftrightarrow$

Correlations between actions of periodic orbits

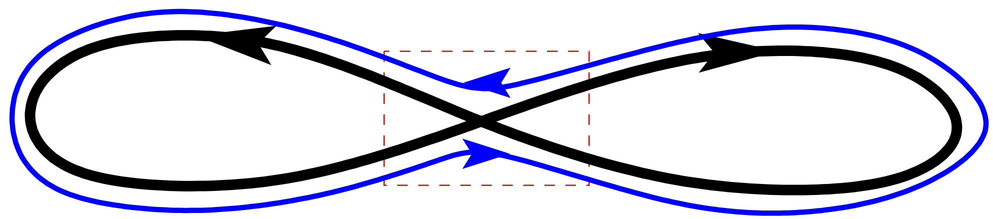
Classical origins of universality

$$K(\tau) = c_1\tau + c_2\tau^2 \ldots$$

c_1 – **diagonal approximation** $\gamma = \gamma'$ M. Berry 1985



Diagonal approximation



Sieber−Richter pairs

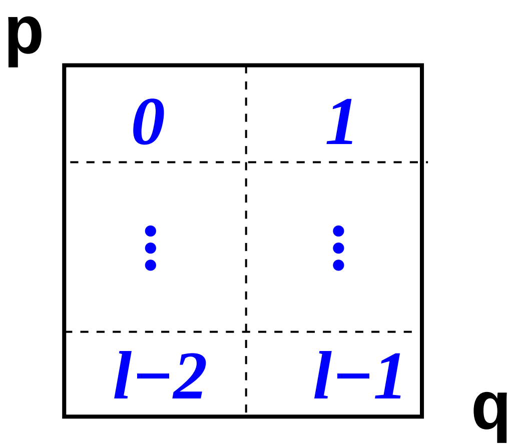
c_2 – **non-trivial correlations** (Sieber-Richter pairs)
M. Sieber K. Richter 2001

$S_\gamma - S_{\gamma'} \sim \hbar \Longrightarrow$ Duration of encounter $\sim \underbrace{\tau_E = \lambda^{-1}|\log\hbar|}_{\text{Ehrenfest time}}$

All orders in $\tau =$ **RMT result** S. Müller, et. al., 2004

Symbolic Dynamics

Continues flow $\Longrightarrow$ Map T (Poincare section)



Phase space partition:

$$V = V_0 \cup V_1 \cup \cdots \cup V_{l-1}$$

Point in the phase space:

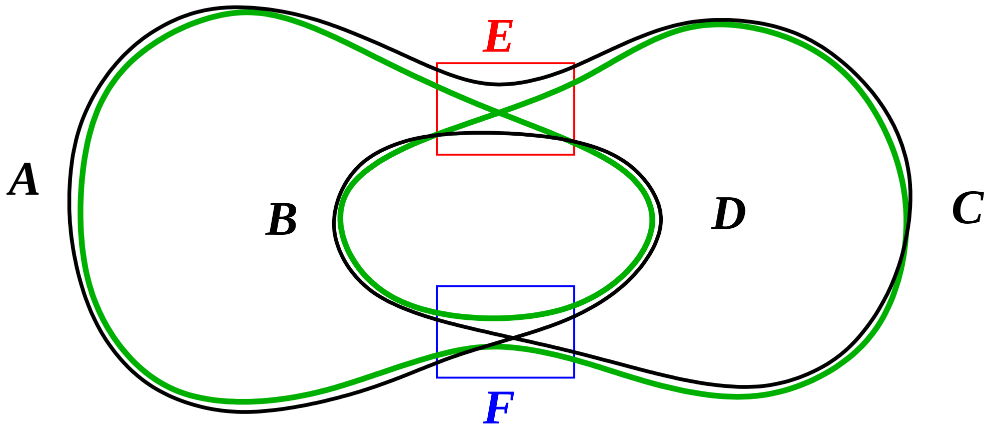
$$x = \underbrace{\ldots x_{-1} x_0}_{\color{blue}{past}} . \underbrace{x_1 x_2 \ldots}_{\color{red}{future}} ; \qquad x_i \in \underbrace{\{0, 1, \ldots l - 1\}}_{alphabet}$$

$$Tx = \ldots x_{-1} x_0 x_1 . x_2 x_3 \ldots$$

Periodic orbits $\Longleftrightarrow$ $[x_1 x_2 \ldots x_n]$

Partner orbits

B. G, V. Osipov 2013



$$[\gamma_1] = [AECFBEDF], \qquad [\gamma_2] = [AEDFBECF]$$

$$E = e_1 e_2 \dots e_p, \qquad F = f_1 f_2 \dots f_p$$

Each p-subsequence of symbols from γ_1 appears in γ_2
Locally similar but not identical $\Longrightarrow$

Two orbits pass approximately the same points of the phase space:

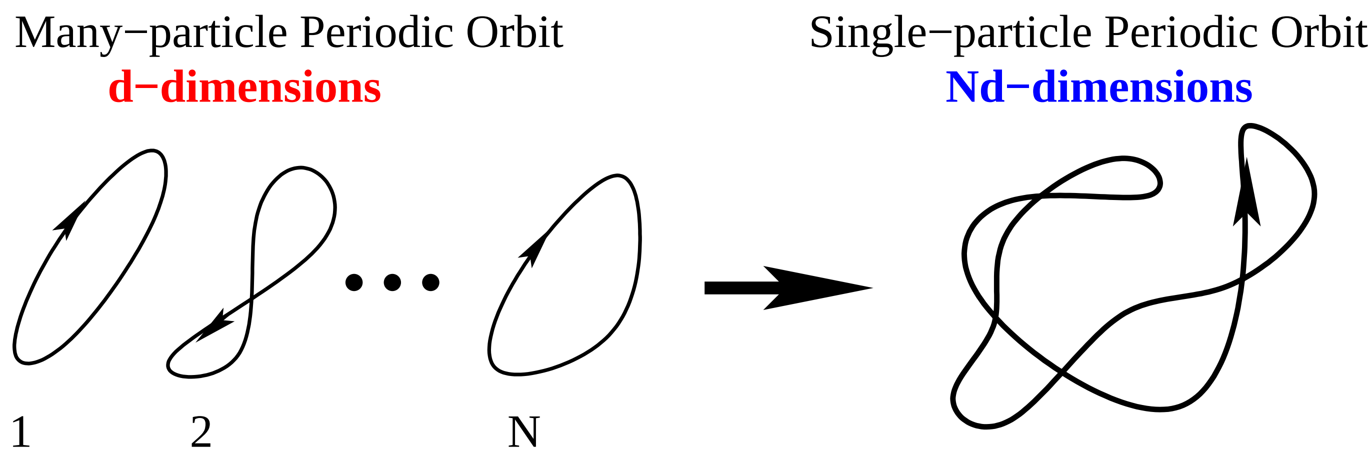
$$\|\gamma_1 - \gamma_2\| \sim \Lambda^{-p}$$

Many-particle systems

$$\mathcal{H} = \sum_{n=1}^{N} \frac{p_n^2}{2m} + V(x_n) + V_{\text{int}}(x_n - x_{n+1})$$

Chaos, Local interactions, Invariance under $n \to n+1$

Two views on dynamics:



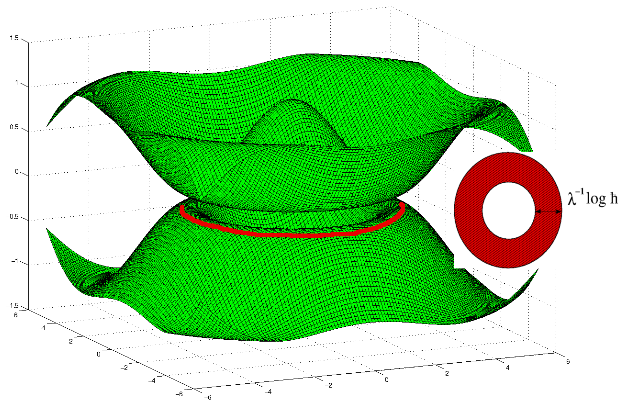
Q: Is the single-particle theory of Quantum Chaos applicable?

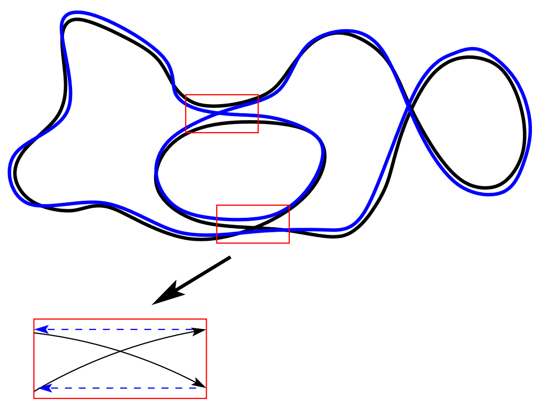
Semiclassical "Field Theory"

Continuous limit: $n \to \eta \in [0, \ell]$**,** $x_{n,t} \to \phi(\eta, t)$

$$\mathcal{L} = \sum_{n=1}^{N} \frac{\dot{q}_{n,t}^2}{2m} + \kappa (x_{n,t} - x_{n+1,t})^2 - V(x_{n,t}) \quad \Longrightarrow$$

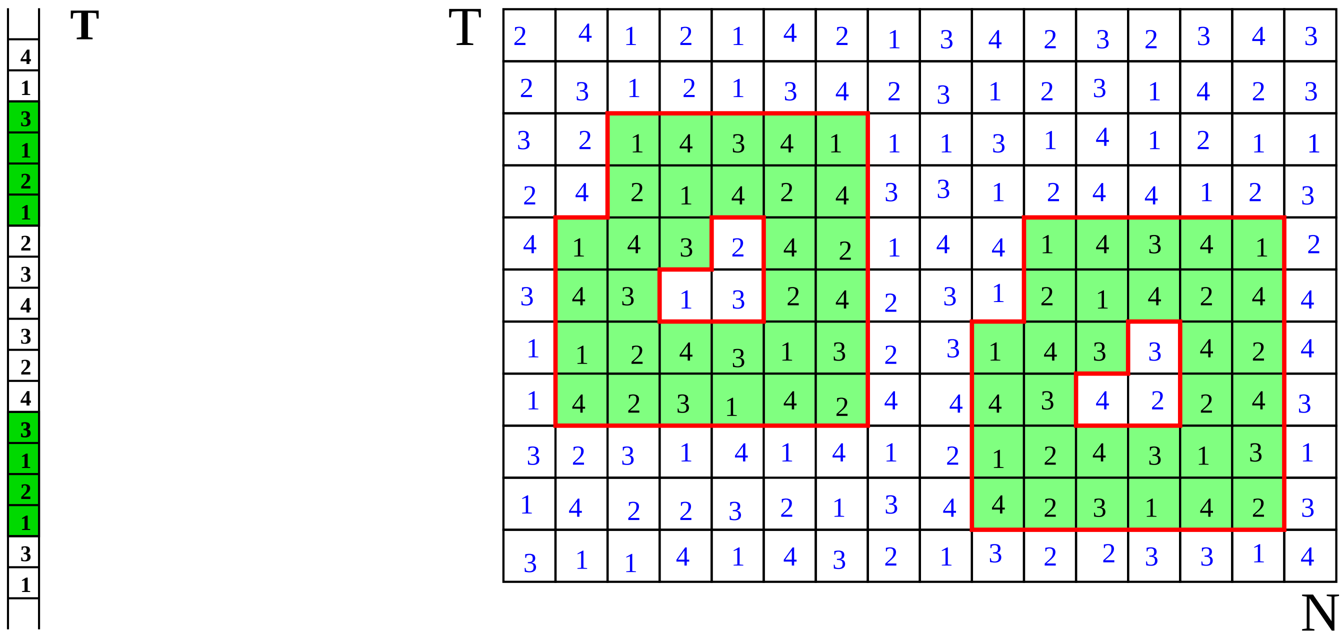
$$\mathcal{L} = \int_0^\ell d\eta \ (\partial_t \phi(\eta, t))^2 + (\partial_\eta \phi(\eta, t))^2 - V(\phi(\eta, t))$$





1) **PO** -are **2D toric surfaces** in d-dim space (Rather than 1D lines in $N \cdot d$-dim)
2) **Encounters** are **"rings"** (Rather than 1D stretches) of "width" $\sim \lambda^{-1} |\log \hbar_{eff}|$

2D Symbolic Dynamics



1) **Small alphabet** (does not grow with N)

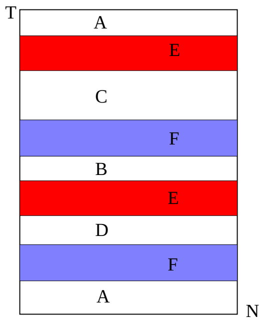
2) **Uniqueness:** Each PO Γ is uniquely encoded by $\mathbb{M}_\Gamma$

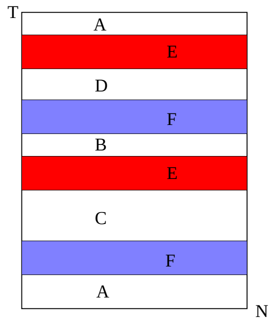
3) **Locality:** $r \times r$ square of symbols around (n, t) defines position of the n'th particle at the time t up to error $\sim \Lambda^{-r}$

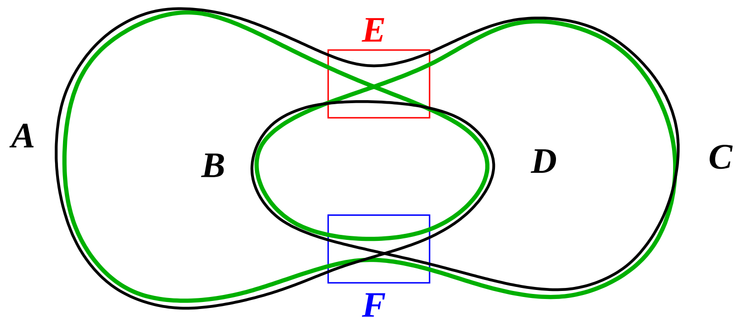
Encounter - repeating region of symbols

Different types of Partner Orbits

A. Single particle partners:



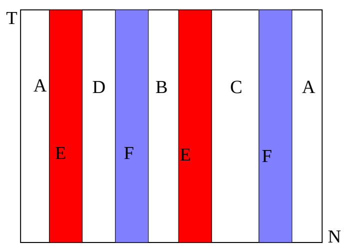


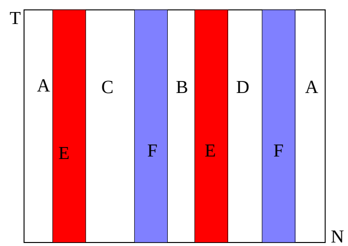


Dominant iff $T \gtrsim W_\hbar \gtrsim N$ - **Single particle theory**

$$W_\hbar \sim \Lambda^{-1} |\log \hbar_{eff}| \approx \text{Width of encounter}$$

B. Dual partners:

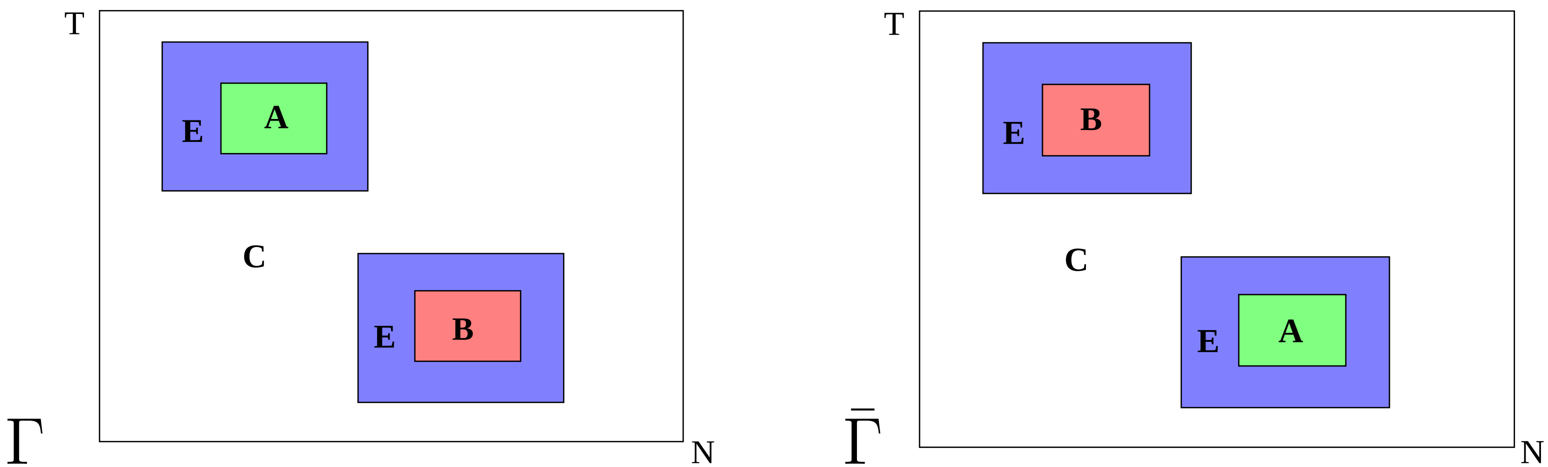




Dominant iff $T \lesssim W_\hbar \lesssim N$ - **Thermodynamic, short time regime**

Different types of Partner Orbits

C. If $T \gtrsim W_\hbar$, $N \gtrsim W_\hbar$ i.e. T and N are larger then **“Ehrenfest scale”**:

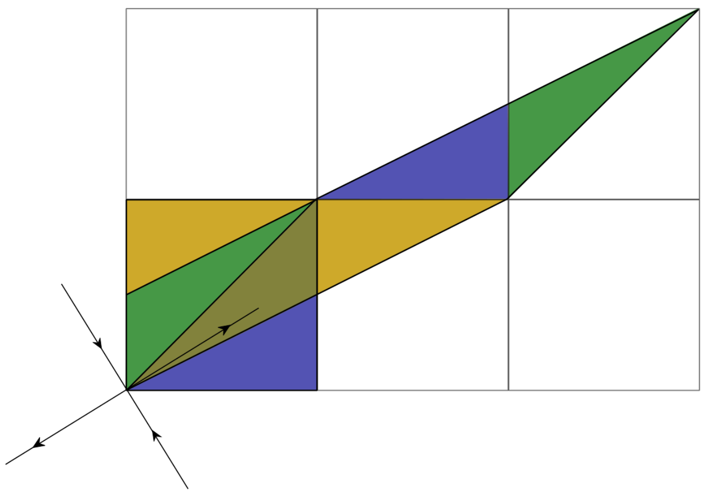


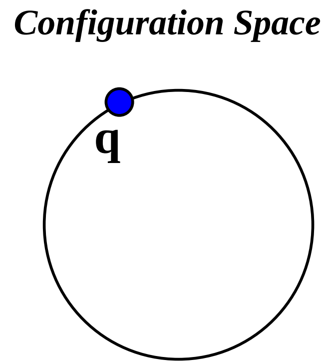
Note: One encounter is enough, even if time reversal symmetry is broken

B, C - Genuine many-particle Quantum Chaos!

A Lone Cat Map: $\mathbb{T}^2 \to \mathbb{T}^2$

Phase space: $q_t, p_t \in [0,1)$, **windings** $\mathrm{m}_t = (m_t^q, m_t^p) \in \mathbb{Z}$



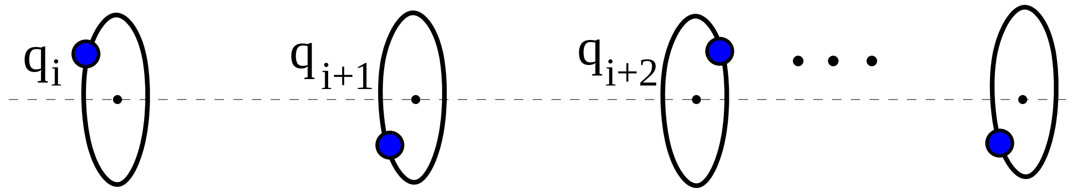


$$\left(\begin{array}{c} q_{t+1} \\ p_{t+1} \end{array}\right) = \left(\begin{array}{cc} a & 1 \\ ab-1 & b \end{array}\right) \left(\begin{array}{c} q_t \\ p_t \end{array}\right) - \left(\begin{array}{c} \mathrm{m}_t^q \\ \mathrm{m}_t^p \end{array}\right),$$

$a, b \in \mathbb{Z}$. Chaos if $|a+b| > 2$

Newton form: $\Delta q_t \equiv q_{t+1} - 2q_t + q_{t-1} = (a+b-2)q_t - \mathrm{m}_t$

Coupled-Cat Maps: $T^{2N} \to T^{2N}$



$$S(\mathsf{q}_t, \mathsf{q}_{t+1}) = S_0(\mathsf{q}_t, \mathsf{q}_{t+1}) + S_{\text{int}}(\mathsf{q}_t),\ \mathsf{q}_t = (q_{1,t}, q_{2,t} \ldots q_{N,t})$$

N Interacting cat maps, $q_{n,t}, p_{n,t} \in [0,1)$:

$$S_0 = \sum_{n=1}^{N} S_{\text{cat}}(q_{n,t}, q_{n,t+1}) + V(q_{n,t}); \qquad S_{\text{int}} = \underbrace{-\sum_{n=1}^{N} q_{n,t} q_{1+n,t}}_{\textit{interactions}}$$

Equations of motion:

$$p_{n,t} = -\frac{\partial S}{\partial q_{n,t}} \qquad p_{n,t+1} = \frac{\partial S}{\partial q_{n,t+1}}$$

Classical Particle-time Duality

Newtonian form:

$$\Delta q_{n,t} = (a+b-4)q_{n,t} + V'(q_{n,t}) - \mathrm{m}_{n,t}$$

Discrete Laplacian:

$\Delta f_{n,t} \equiv f_{n+1,t} + f_{n-1,t} + f_{n,t+1} + f_{n,t-1} - 4f_{n+1,t}$

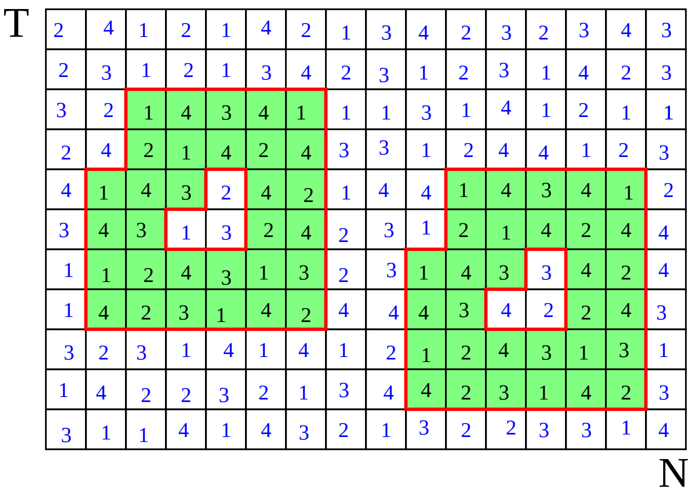
Particle-time symmetry: $t \longleftrightarrow n$ $\Longrightarrow$

N-particle POs $\{\Gamma\}$ of period $T \Longleftrightarrow T$-particle POs $\{\Gamma'\}$ of period N

$$S(\Gamma) = S(\Gamma'), \qquad A_\Gamma = A_{\Gamma'}$$

$\{\mathrm{m}_{n,t}\}$ - provide symbolic encoding of POs

2D Symbolic Dynamics



$$\mathbb{M}_\Gamma = \begin{pmatrix} \mathrm{m}_{1,1} & \mathrm{m}_{2,1} & \dots & \mathrm{m}_{N,1} \\ \mathrm{m}_{1,2} & \mathrm{m}_{2,2} & \dots & \mathrm{m}_{N,2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{m}_{1,T} & \mathrm{m}_{2,T} & \dots & \mathrm{m}_{N,T} \end{pmatrix}$$

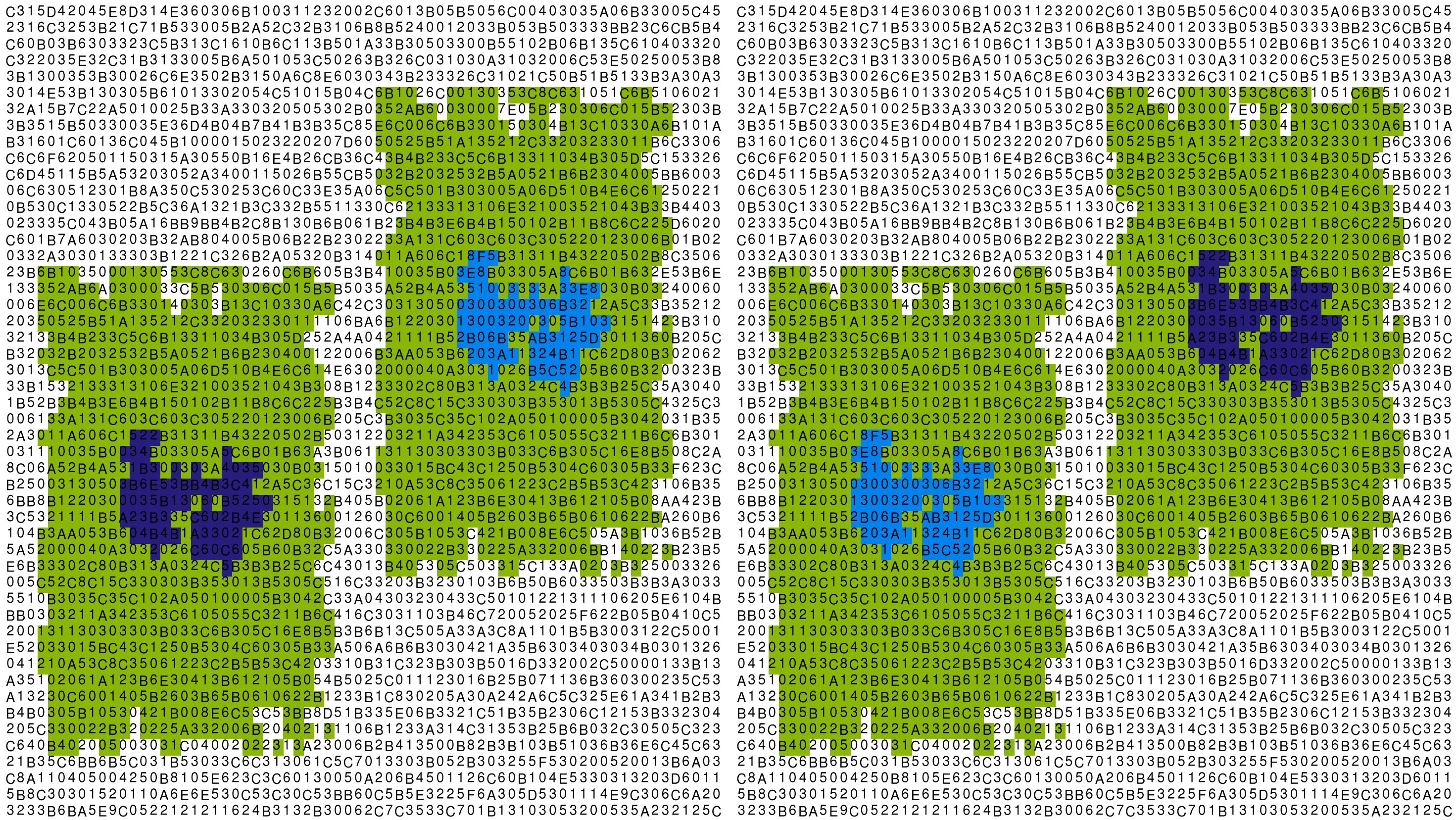
$\surd$ **Small alphabet** (does not grow with N)

$\surd$ **Uniqueness** + $\underline{\Gamma \text{ can be easily restored from } \mathbb{M}_\Gamma}$

$\surd$ **Locality** ($r \times r$ square of symbols around (n, t) defines approx. position of the n'th particle at the time t)
B.G. V. Osipov (2015),
B.G., L Han, R. Jafari, A. K. Saremi, P Cvitanović (2016)

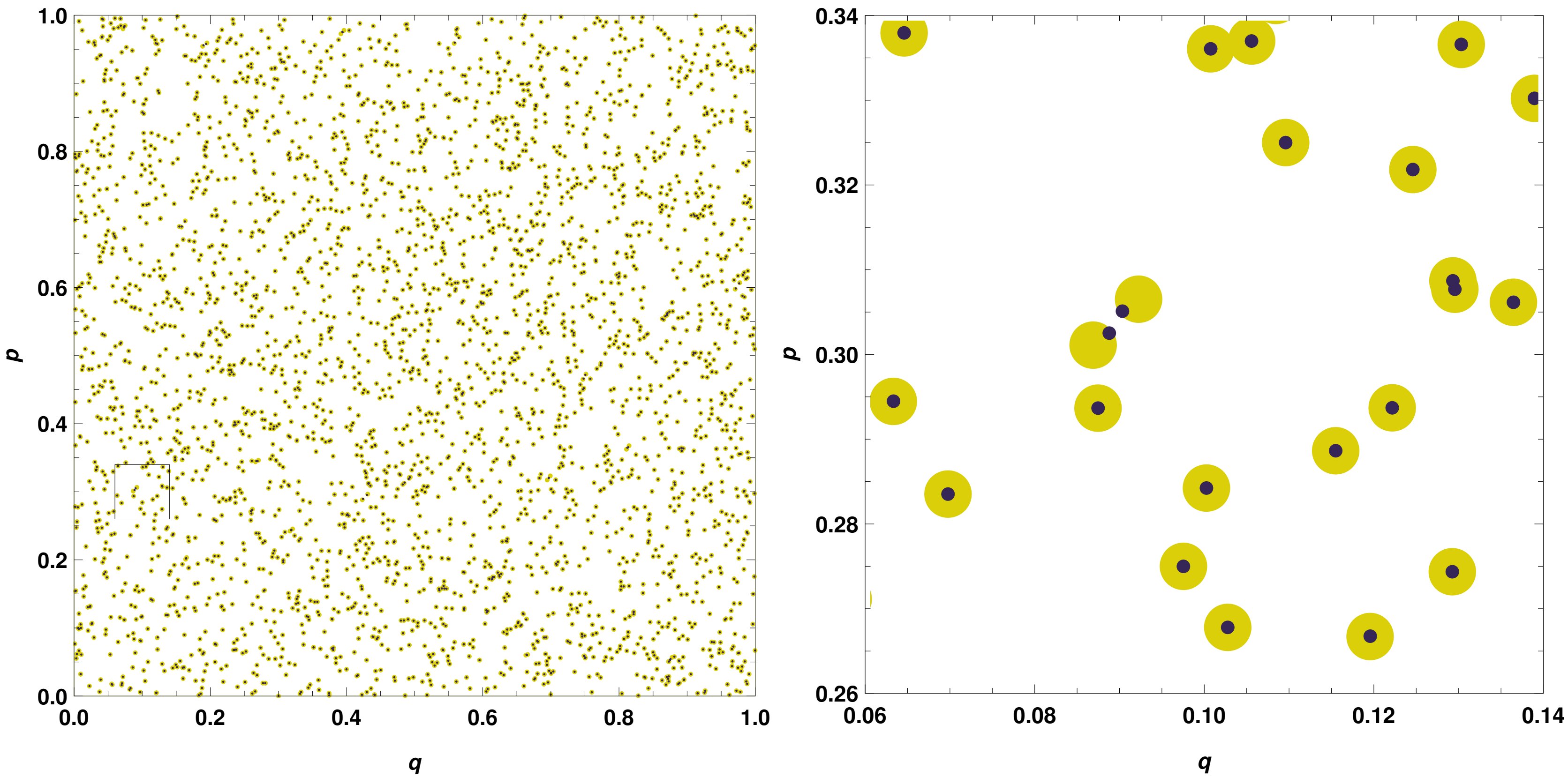
Example of Partner Orbits

$$T = 50,\ N = 70,\ a = 3,\ b = 2$$



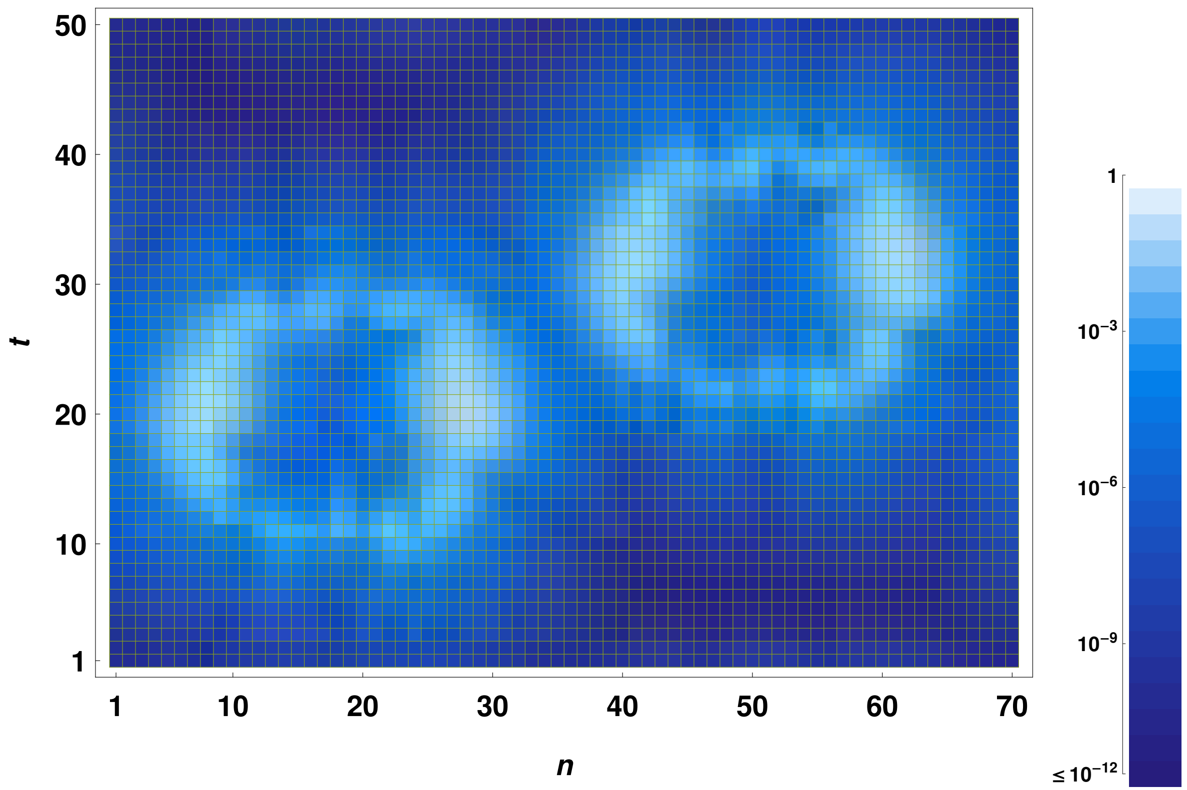
m	$\binom{0}{0}$	$\binom{0}{1}$	$\binom{1}{1}$	$\binom{1}{2}$	$\binom{1}{3}$	$\binom{2}{3}$	$\binom{2}{4}$	$\binom{2}{5}$	$\binom{3}{5}$	$\binom{3}{6}$	$\binom{0}{-1}$	$\binom{-1}{-1}$	$\binom{-1}{-2}$	$\binom{-1}{-3}$	$\binom{-2}{-3}$	$\binom{-2}{-4}$
a	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

Example of Partner Orbits



All the points of $\Gamma = \{(q_{n,t}, p_{n,t})\}$ and $\bar{\Gamma} = \{(\bar{q}_{n,t}, \bar{p}_{n,t})\}$ are paired

Distances between paired points



$$d_{n,t} = \sqrt{(q_{n,t} - \bar{q}_{n',t'})^2 + (p_{n,t} - \bar{p}_{n',t'})^2},$$

Largest distances $\sim 2 \cdot 10^{-3}$ are between points in encounters

Quantisation

Hannay, Berry (1980); Keating (1991)

U_N is $L^N \times L^N$ unitary matrix, $L = \hbar_{eff}^{-1}$

Translational symmetries: $\Longrightarrow$ N subspectra approximately of the same size $= L^N/N$

Gutzwiller trace formula

Rivas, Saraceno, A. de Almeida (2000)

$$\mathrm{Tr}\,(U_N)^T = \left|\det(\mathcal{B}_N^T - 1)\right|^{-\frac{1}{2}} \sum_{\Gamma \in \mathrm{PO}} \exp(-i2\pi L S_\Gamma).$$

All entries are symmetric under exchange $N \leftrightarrow T$

Quantum Duality

$$\mathrm{Tr}\,(U_N)^T = \mathrm{Tr}\,(U_T)^N$$

Form Factor: $K_N(T) = \dfrac{1}{2L^N}\left\langle \left|\mathrm{Tr}\,(U_N)^T\right|^2 \right\rangle$

For short times $T < n_E = \lambda^{-1}\log L$, $\quad N \sim L^T$
Regime dual to universal:

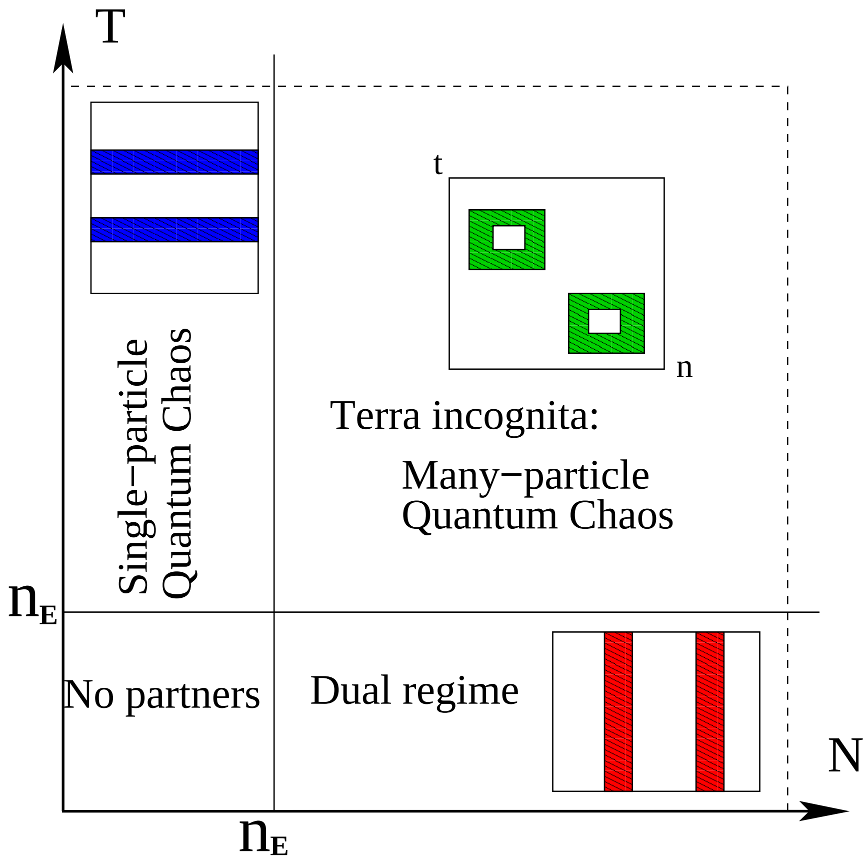
$$K_N(T) = L^{T-N} K_\beta(TN/L^T)$$

In particular for very short times $L^T/T < N$, $\quad K_\beta \approx 1$

$$K_N(T) \approx L^T/L^N$$

Short time exponential growth instead of linear TN/L^N

Summary



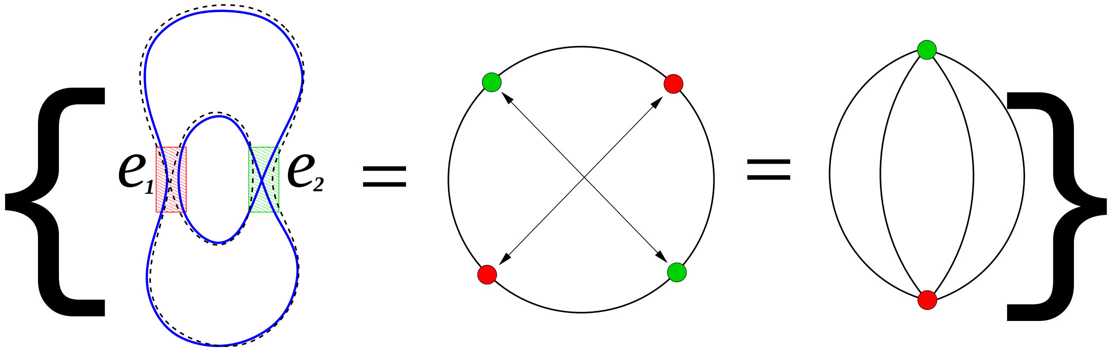
$$\mathcal{K} = \frac{1}{TN} \left\langle \left| \mathrm{Tr}\, (U_N)^T \right|^2 \right\rangle$$

Duality:

$$\mathcal{K}(N,T) = \mathcal{K}(T,N)$$

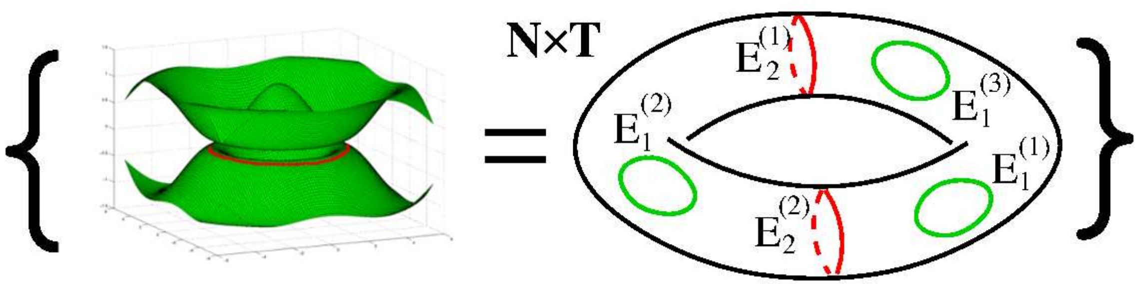
Many-particle Semiclassical Programm

Single-particle structure diagrams:



Distinguished by order of encounters

Many-particle structure diagrams:



Distinguished by order and winding numbers ω of encounters!